On minimal finite factor groups of outer automorphism groups of free groups

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Abstract

We prove that, for $n = 3$ and 4, the minimal nonabelian finite factor group of the outer automorphism group $\mathrm{Out}\, F_n$ of a free group of rank n is the linear group $\mathrm{PSL}_n(\mathbb{Z}_2)$ (conjecturally, this may remain true for arbitrary rank $n > 2$). We also discuss some computational results on low index subgroups of $\mathrm{Aut}\, F_n$ and $\mathrm{Out}\, F_n$, for $n = 3$ and 4, using presentations of these groups.

1 Introduction

It is shown in [Z] that the minimal nontrivial finite quotient (nontrivial factor group of smallest possible order) of the mapping class group $\mathcal{M}_g$ of a closed orientable surface of genus g is the symplectic group $\mathrm{PSp}_{2g}(\mathbb{Z}_2)$, for $g = 3$ and 4; in fact, this may remain true for arbitrary genus $g \geq 3$. Since $\mathcal{M}_g$ is perfect for $g \geq 3$, such a minimal nontrivial finite quotient is a nonabelian simple group. There are canonical projections onto symplectic groups

$$\mathcal{M}_g \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}_p) \to \mathrm{PSp}_{2g}(\mathbb{Z}_p),$$

and the projective symplectic groups $\mathrm{PSp}_{2g}(\mathbb{Z}_p)$ are simple if p is prime and $g \geq 3$. It is a consequence of the congruence subgroup property for the symplectic groups $\mathrm{Sp}_{2g}(\mathbb{Z})$ that their finite simple quotients are exactly the finite projective symplectic groups $\mathrm{PSp}_{2g}(\mathbb{Z}_p)$ (see [Z]). But also for the mapping class groups $\mathcal{M}_g$, all known finite quotients seem to be strongly connected to the symplectic groups, and it would be interesting to know what other finite simple groups can occur (see [T] for a computational approach for genus two and three, and [MR] for comments on the congruence subgroup property).

In the present note, we consider the outer automorphism group $\mathrm{Out}\, F_n$ of a free group F_n of rank n. It is well-known that $\mathrm{Out}\, F_2 \cong \mathrm{GL}_2(\mathbb{Z}) \cong \mathbb{D}_{12} *_{\mathbb{D}_4} \mathbb{D}_8$ (a free product with amalgamation of two dihedral groups of orders 12 and 8), and we shall assume in the following that $n \geq 3$; then the abelianization of $\mathrm{Out}\, F_n$ has order two. There is a canonical projection of $\mathrm{Out}\, F_n$ onto $\mathrm{GL}_n(\mathbb{Z})$, and we

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consider also the preimage of $\mathrm{SL}_n(\mathbb{Z})$ in $\mathrm{Out}\, F_n$ which we denote by $\mathrm{SOut}\, F_n$ (the unique subgroup of index two of $\mathrm{Out}\, F_n$). It is well-known that $\mathrm{SOut}\, F_n$ is a perfect group (see [Ge] for a presentation), so the minimal nontrivial quotient will be again a nonabelian simple group. There are projections

$$\mathrm{SOut}\, F_n \to \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_p) \to \mathrm{PSL}_n(\mathbb{Z}_p),$$

and the finite linear groups $\mathrm{PSL}_n(\mathbb{Z}_p)$ are simple if p is prime. It is a consequence of the congruence subgroup property for the linear group $\mathrm{SL}_n(\mathbb{Z})$ that the finite simple quotients of $\mathrm{SL}_n(\mathbb{Z})$ are exactly the finite projective linear groups $\mathrm{PSL}_n(\mathbb{Z}_p)$, p prime ([Z]).

Our main result is the following:

Proposition. *For $n = 3$ and 4, the minimal nontrivial finite quotient of $\mathrm{SOut}\, F_n$, and also the minimal nonabelian finite quotient of $\mathrm{Out}\, F_n$, is the linear group $\mathrm{PSL}_n(\mathbb{Z}_2)$.*

We note that $\mathrm{PSL}_3(\mathbb{Z}_2) \cong \mathrm{PSL}_2(\mathbb{Z}_7)$ is the unique simple group of order 168, and that $\mathrm{PSL}_4(\mathbb{Z}_2)$ is isomorphic to the alternating group $\mathbb{A}_8$, of order 20160. Conjecturally, the Proposition remains true for arbitrary $n \geq 3$. Concerning other finite simple groups, it is shown in [Gi] that infinitely many alternating groups occur as quotients of $\mathrm{Out}\, F_n$.

As a consequence of the Proposition we have also the following:

Corollary. *The minimal index of a proper subgroup of $\mathrm{SOut}\, F_4$, and also of a proper subgroup of $\mathrm{Out}\, F_4$ different from $\mathrm{SOut}\, F_4$, is eight (the minimal index of a proper subgroup of $\mathrm{PSL}_4(\mathbb{Z}_2) \cong \mathbb{A}_8$).*

For $n = 3$ this minimal index should be seven (the minimal index of a proper subgroup of $\mathrm{PSL}_3(\mathbb{Z}_2) \cong \mathrm{PSL}_2(\mathbb{Z}_7)$) but for the moment we cannot exclude index six by the present methods; we verified this, however, by computational methods (GAP), see section 3 for some comments.

2   Proof of the Proposition and the Corollary

We denote by $\mathrm{Aut}\, F_n$ the automorphism group of the free group F_n and by $\mathrm{SAut}\, F_n$ its subgroup of index two which is the preimage of $\mathrm{SL}_n(\mathbb{Z})$ under the canonical projection of $\mathrm{Aut}\, F_n$ onto $\mathrm{GL}_n(\mathbb{Z})$. Fixing a free generating set of F_n, inversions and permutations of generators generate a subgroup (Weyl group) $W_n \cong (\mathbb{Z}_2)^n \rtimes \mathbb{S}_n$ of $\mathrm{Aut}\, F_n$; let SW_n denote $W_n \cap \mathrm{SAut}\, F_n$, with $SW_n \cong (\mathbb{Z}_2)^{n-1} \rtimes \mathbb{S}_n$.

We note that, by results in [WZ], W_n is the finite subgroup of maximal possible order of both $\operatorname{Aut} F_n$ and $\operatorname{Out} F_n$, for $n \geq 3$, unique up to conjugation if $n > 3$ (for $n = 3$ there is one other subgroup of maximal possible order 48).

Let Δ denote the central element of W_n inverting all generators; note that Δ is in SW_n if and only if n is even.

The proof of the Proposition is based on the following:

Lemma ([BV, Prop. 3.1]). *Let $n \geq 3$ and ϕ be a homomorphism from* $\operatorname{SAut} F_n$ *to a group G. If the restriction of ϕ to SW_n has nontrivial kernel K then one of the following holds:*

i) *n is even, $K = \langle \Delta \rangle$ and ϕ factors through $\operatorname{PSL}_n(\mathbb{Z})$;*

ii) *K is the intersection of SW_n with the subgroup $(\mathbb{Z}_2)^n$ of W_n generated by all inversions and the image of ϕ is isomorphic to $\operatorname{PSL}_n(\mathbb{Z}_2)$, or*

iii) *ϕ is the trivial map.*

Proof of the Proposition. We consider the case $n = 3$ first. Since $\operatorname{SOut} F_3$ is perfect, a minimal nontrivial finite quotient of $\operatorname{SOut} F_3$ is a nonabelian simple group. The only nonabelian simple group with an order smaller than the order 168 of the linear group $\operatorname{PSL}_3(\mathbb{Z}_2) \cong \operatorname{PSL}_2(\mathbb{Z}_7)$ is the alternating group $\mathbb{A}_5$, of order 60 (see [C] for information about the finite simple groups). Since the order 24 of SW_3 does not divide the order of $\mathbb{A}_5$, by the Lemma every homomorphism from $\operatorname{SOut} F_3$ to $\mathbb{A}_5$ is trivial, hence the minimal possibility for a nontrivial finite quotient of $\operatorname{SOut} F_3$ is the linear group $\operatorname{PSL}_3(\mathbb{Z}_2) \cong \operatorname{PSL}_2(\mathbb{Z}_7)$, the unique simple group of order 168.

As for $\operatorname{Out} F_3$, if the finite nonabelian group G is a quotient of $\operatorname{Out} F_3$ then the image of $\operatorname{SOut} F_3$ has index one or two in G and order at least 168. Since $\operatorname{Out} F_3$ surjects onto $\operatorname{PSL}_3(\mathbb{Z}_2) = \operatorname{PGL}_3(\mathbb{Z}_2)$, of order 168, this is again the minimal possibility for G.

We come now to the proof of the Proposition for $n = 4$. Suppose that $\phi : \operatorname{SOut} F_4 \to G$ is a nontrivial homomorphism onto a finite simple group G of order less than the order 20160 of $\operatorname{PSL}_4(\mathbb{Z}_2) \cong \mathbb{A}_8$. The simple groups of order less than 20160 are the following (see [C]):

- the alternating groups $\mathbb{A}_d$ of degrees $d = 5, 6$ or 7;

- the linear groups $\operatorname{PSL}_2(\mathbb{Z}_p) = \operatorname{L}_2(p)$, for the primes $p = 7, 11, 13, 17, 19, 23, 29$ and 31;

- the linear groups $\operatorname{PSL}_2(q) = \operatorname{L}_2(q)$, for the prime powers $q = 8, 9, 16, 25$ and 27 (over the finite fields of the corresponding orders);

- the linear group $\operatorname{PSL}_3(\mathbb{Z}_3) = \operatorname{L}_3(3)$;

- the unitary group $\mathrm{PSU}_3(\mathbb{Z}_3) = \mathrm{U}_3(3)$;

- the Mathieu group M_{11}.

Since the order of none of these groups is divided by the order $2^6 \cdot 3$ of SW_4, the restriction of ϕ to SW_4 has nontrivial kernel and the Lemma applies. Since the cases ii) and iii) of the Lemma are excluded by the hypotheses, we are necessarily in case i). Now also case i) may be excluded by appealing to the fact that the finite quotients of $\mathrm{PSL}_4(\mathbb{Z})$ are exactly the groups $\mathrm{PSL}_4(\mathbb{Z}_p)$, p prime ([Z, Theorem 1]); however also the following more direct argument, in the spirit of the previous ones, applies. In case i) of the Lemma, the kernel of the restriction of ϕ to SW_4 is the subgroup of order two generated by the central involution Δ of SW_4. Hence the factor group $SW_4/\langle\Delta\rangle$, of order $2^5 \cdot 3$, embeds into G. However none of the above groups has such a subgroup: considering orders again, the only candidates which remain are $\mathrm{U}_3(3)$ and $\mathrm{L}_2(31)$, but both do not have a subgroup isomorphic to $SW_4/\langle\Delta\rangle$.

There is one other simple group of order 20160 not isomorphic to $\mathrm{PSL}_4(\mathbb{Z}_2) \cong \mathbb{A}_8$, the linear group $\mathrm{PSL}_3(4) = \mathrm{L}_3(4)$. Considering the maximal subgroups of $\mathrm{L}_3(4)$, it is easy to see that $G = \mathrm{L}_3(4)$ has no subgroup isomorphic to SW_4, so the Lemma applies again leaving us with case i); finally, also case i) is excluded since $\mathrm{L}_3(4)$ has no subgroup isomorphic to $SW_4/\langle\Delta\rangle$ (or by appealing again to [Z, Theorem 1]).

This completes the proof of the Proposition for $\mathrm{SOut}\, F_4$, and hence also for $\mathrm{Out}\, F_4$.

Proof of the Corollary. Both $\mathrm{SOut}\, F_4$ and $\mathrm{Out}\, F_4$ have subgroups of index eight since both admit surjections onto the alternating group $\mathbb{A}_8 \cong \mathrm{PSL}_4(\mathbb{Z}_2)$. By the proof of the Proposition, $\mathrm{SOut}\, F_4$ does not admit a nontrivial homomorphism to an alternating groups $\mathbb{A}_d$ of degree $d < 8$, hence $\mathrm{SOut}\, F_4$ has no proper subgroup of index less than eight. Similarly, the same holds for $\mathrm{Out}\, F_4$ which does not admit a nontrivial homomorphism to a symmetric group $\mathbb{S}_d$ of degree $d < 8$.

3   Comments on computations

We employed the low index subgroup procedure of the computer algebra system GAP in order to find the smallest indices of proper subgroups of $\mathrm{SAut}\, F_3$ and $\mathrm{SOut}\, F_3$. We used the 4-generator presentation of $\mathrm{Aut}\, F_3$ in [MaKS, section 3.5, Corollary N1] (note that some of the relations apply only for $n > 3$; see also [CM, section 7.3], and [MC], [NN, section 6], section 6] for presentations of $\mathrm{Out}\, F_3$), and created by GAP the unique subgroups $\mathrm{SAut}\, F_3$ and $\mathrm{SOut}\, F_3$ of index two. We found that the three smallest indices of a proper subgroup

of SAut F_3 and SOut F_3 are 7, 8 and 13; the factor groups of the cores of the corresponding subgroups (the largest normal subgroup contained in a subgroup) are $PSL_3(\mathbb{Z}_2) = GL_3(\mathbb{Z}_2)$ for indices 7 and 8, a semidirect product $(\mathbb{Z}_2)^3 \rtimes GL_3(\mathbb{Z}_2)$ for index 8, and $PSL_3(\mathbb{Z}_3)$ for index 13.

Remark. Concerning the smallest indices of proper subgroups of $SL_3(\mathbb{Z})$, index 8 is now missing and there remain only the indices 7 and 13 (see [MaKS, section 3.5] for a presentation of $GL_r(\mathbb{Z})$). Computing the abelianization of the core of the index 8 subgroup of SOut F_3 (a normal subgroup of index 1344, with quotient $(\mathbb{Z}_2)^3 \rtimes GL_3(\mathbb{Z}_2)$), we found the free abelian group $\mathbb{Z}^{14}$; so this gives an explicit example of a finite index subgroup of SOut F_3 with infinite abelianization. We note that the existence of such subgroups was known since by [MC], Out F_3 is virtually residually torsion-free nilpotent; on the other hand, no such examples seem to be known for rank $n > 3$ (see [T] for the corresponding situation for mapping class groups).

We then went on to compute that the three minimal indices of subgroups of SAut F_4 are 8, 15 and 16; the factor groups of the cores of the corresponding subgroups are $PSL_4(\mathbb{Z}_2) = GL_4(\mathbb{Z}_2) \cong \mathbb{A}_8$ for indices 8 and 15 and, for index 16, a semidirect product $(\mathbb{Z}_2)^4 \rtimes GL_4(\mathbb{Z}_2)$ of order 322560; at present we don't know if the abelianization of the core in the case of index 16 is finite or infinite. On the other hand, the minimal indices of subgroups of SOut F_4 are 8 and 15, and index 16 is now missing. We note that each SAut F_r admits a surjection onto a finite group with a normal subgroup $(\mathbb{Z}_n)^r$ and factor group $GL_r(\mathbb{Z}_n)$ (by dividing out first the kernel of the natural projection of Inn $F_r \cong F_r$ onto $(\mathbb{Z}_n)^r$, then projecting onto $GL_r(\mathbb{Z}_n)$).

Employing in addition the quotient group procedure of GAP, we verified also that the three smallest simple factor groups of SAut F_3 and SOut F_3 are, as expected, the groups $PSL_3(\mathbb{Z}_2)$ of order 168, $PSL_3(\mathbb{Z}_3)$ of order 5616, and $PSL_3(\mathbb{Z}_5)$ of order 372000 (we note e.g. that, by the Lemma, the first Janko group does not occur since it has no subgroup isomorphic to $SW_3 \cong S_4$).

We note that these computations can be slightly extended but that already the suspected minimal index 31 of a proper subgroup of SOut F_5 as well as its suspected minimal nonabelian quotient $PSL_5(\mathbb{Z}_2)$ (of order 9.999.360, with a subgroup of index 31) appear quite large for such computations so we didn't pursue this further.

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